Convergence analyses of the Peaceman–Rachford and Douglas–Rachford Schemes for Semilinear Evolution Equations

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Semilinear evolution equation

Evolution equation on a Hilbert space $\mathcal{H}$:

$$\dot{u} = (A + F)u, \quad u(0) = \eta$$

A linear (typically diffusion operator)
F nonlinear (typically reaction operator)

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Example

Gray–Scott pattern formation model

$$\begin{cases} \dot{u}_1 = \Delta u_1 - u_1 u_2^2 + (1 - u_1) \\ \dot{u}_2 = \Delta u_2 + u_1 u_2^2 - u_2 \end{cases}$$

$$A : \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix}, \qquad F : \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} -u_1 u_2^2 + (1 - u_1) \\ u_1 u_2^2 - u_2 \end{pmatrix}$$

Splitting idea

Full problem

$$\dot{u} = (A + F)u, \quad u(0) = \eta$$

difficult, expensive

Splitting idea: Instead consider the subproblems

$$\dot{u} = Au \qquad \text{and} \qquad \dot{u} = Fu$$

simple, cheap

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Splitting approximation

$$\text{One time step: } u(k) = \mathrm{e}^{k(A+F)}\eta \approx \mathrm{e}^{kF}\mathrm{e}^{kA}\eta$$

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Splitting approximation

One time step: $\quad u(k) = \mathrm{e}^{k(A+F)}\eta \approx \mathrm{e}^{kF}\mathrm{e}^{kA}\eta$

n time steps: $\quad u(nk) = \mathrm{e}^{nk(A+F)}\eta \approx (\mathrm{e}^{kF}\mathrm{e}^{kA})^n\eta$

Some splitting methods

Subproblems

$$\dot{u} = Au \qquad \text{and} \qquad \dot{u} = Fu$$

Splitting approximation

$$u(nk) = \mathrm{e}^{nk(A+F)}\eta \;\approx\; S^n\eta$$

Splitting methods, examples

Lie–Trotter: $S = \mathrm{e}^{kF}\mathrm{e}^{kA}$

(Full) Lie: $S = (I - kF)^{-1}(I - kA)^{-1}$

Peaceman–Rachford: $S = (I - \frac{k}{2}F)^{-1}(I + \frac{k}{2}A)(I - \frac{k}{2}A)^{-1}(I + \frac{k}{2}F)$

Douglas–Rachford: $S = (I - kF)^{-1}(I - kA)^{-1}(I + k^2 AF)$

[see e.g. Hundsdorfer & Verwer 2003]

Dissipative operators

$\mathcal{H}$ Hilbert space

$G : \mathcal{D}(G) \subset \mathcal{H} \to \mathcal{H}$ is maximal dissipative iff there is a constant $M[G] \geq 0$ s.t.

$$
\begin{aligned}
&(Gu - Gv,\ u - v) \leq M[G] \, \|u - v\|^2 \\
&\mathcal{R}(I - kG) = \mathcal{H}
\end{aligned}
$$

for all $u, v \in \mathcal{D}(G)$ and $k \in [0, 1/M[G])$.

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for all $u, v \in \mathcal{D}(G)$ and $k \in [0, 1/M[G])$.

Example

Laplace, dissipativity:

$$(\Delta u, u)_{L^2(\Omega)} = -(\nabla u, \nabla u)_{L^2(\Omega)} = -\left\| \nabla u \right\|^2_{L^2(\Omega)} \leq 0,$$
$$\forall u \in H^2(\Omega) \cap H^1_0(\Omega)$$

Maximality a bit more involved

Dissipative operators

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$G : \mathcal{D}(G) \subset \mathcal{H} \to \mathcal{H}$ is maximal dissipative iff there is a constant $M[G] \geq 0$ s.t.

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$$\mathcal{R}(I - kG) = \mathcal{H}$$

for all $u, v \in \mathcal{D}(G)$ and $k \in [0, 1/M[G])$.

Theorem

Resolvent $(I - kG)^{-1}$ exists and

$$L[(I - kG)^{-1}] \leq 1/(1 - kM[G])$$

[see e.g. Barbu 2010]

Dissipative evolution equations

Evolution equation

$$\dot{u} = (A + F)u, \quad u(0) = \eta$$

Assumption

$A + F$ is maximal dissipative on $\mathcal{H}$

Theorem

If $\eta \in \overline{\mathcal{D}(A + F)}$ there is a unique mild solution

$$u(t) = \mathrm{e}^{t(A+F)}\eta := \lim_{n\to\infty}(I - \frac{t}{n}(A + F))^{-n}\eta$$

[see e.g. Barbu 2010]

Semilinear setting – convergence theorem

Assumption

A, F, (and $A+F$) are maximal dissipative on $\mathcal{H}$

Assumption

$u^{(3)}, A\ddot{u}, A^2\dot{u} \in L^1(0,T;\mathcal{H})$

$f \in L^1(0,T;\mathcal{H})$ means $\int_0^T \|f(t)\|_{\mathcal{H}}\,\mathrm{d}t < \infty$

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Theorem

For $S = (I - \frac{k}{2}F)^{-1}(I + \frac{k}{2}A)(I - \frac{k}{2}A)^{-1}(I + \frac{k}{2}F)$

$$\|u(nk) - S^n\eta\| \leq Ck^2 \sum_{i=0}^{2} \left\| A^{2-i} u^{(i+1)} \right\|_{L^1(0,T;\mathcal{H})}$$

$nk \leq T$

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A, F, (and $A+F$) are maximal dissipative on $\mathcal{H}$

Assumption

$\ddot{u}, A\dot{u} \in L^1(0,T;\mathcal{H})$

Theorem

For $S = (I-kF)^{-1}(I-kA)^{-1}(I+k^2AF)$

$$\|u(nk) - S^n\eta\| \le Ck\sum_{i=0}^{1}\left\|A^{1-i}u^{(i+1)}\right\|_{L^1(0,T;\mathcal{H})}$$

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Semilinear setting – convergence theorem

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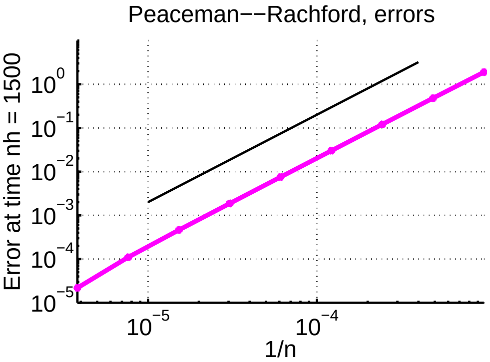
$nk \leq T$

Remark: Always $o(1)$-convergence

Semilinear application: Gray–Scott

Peaceman–Rachford solution of the Gray–Scott equations

$$\begin{cases} \dot{u}_1 = 8 \cdot 10^{-4} \Delta u_1 - u_1 u_2^2 + 0.024(1 - u_1) \\ \dot{u}_2 = 4 \cdot 10^{-4} \Delta u_2 + u_1 u_2^2 - 0.084 u_2 \end{cases}$$



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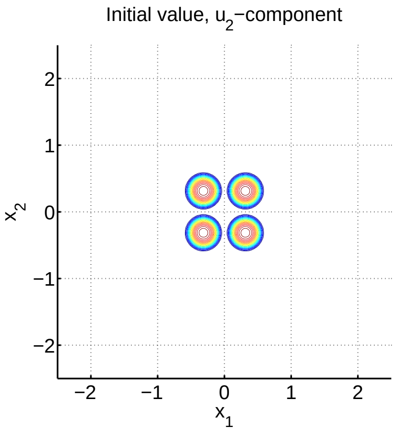

Full discretization – convergence theorem

Assumption

Also $F = B$ is linear. (And $A + B$ is maximal dissipative on $\mathcal{H}$.)

On finite dimensional spaces $\mathcal{H}_h$ define

$$\dot{u}_h = (A_h + B_h)u_h, \quad u_h(0) = \eta_h$$

space discretizations of

$$\dot{u} = (A + B)u, \quad u(0) = \eta$$

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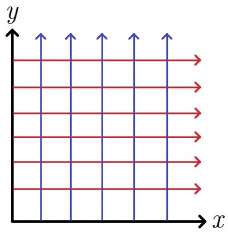
Example

Dimension splitting

$$\dot{u} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u$$



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space discretizations of

$$\dot{u} = (A + B)u, \quad u(0) = \eta$$

Assumption

- *A_h and B_h dissipative on $\mathcal{H}_h$*
- $\left\|(A+B)^{-1}v - (A_h + B_h)^{-1}P_h v\right\| \leq C(v)h^s$
- $\left\|A_h(A_h + B_h)^{-1}\right\|$ *and* $\left\|A_h^2(A_h + B_h)^{-2}\right\|$ *uniformly bounded*
- $\ldots$

Full discretization – convergence theorem

Assumption

Also $F = B$ is linear. (And $A + B$ is maximal dissipative.)

Assumption

- A_h *and* B_h *dissipative*
- $\left\|(A+B)^{-1}v - (A_h + B_h)^{-1}P_h v\right\| \leq C(v)h^s$
- $\left\|A_h(A_h + B_h)^{-1}\right\|$ *and* $\left\|A_h^2(A_h + B_h)^{-2}\right\|$ *uniformly bounded*
- $\ldots$

Theorem

For $S_h = (I - \frac{k}{2}B_h)^{-1}(I + \frac{k}{2}A_h)(I - \frac{k}{2}A_h)^{-1}(I + \frac{k}{2}B_h)$

$$\|u(nk) - S_h^n \eta_h\| \leq C(u)(h^s + k^2)$$

Full discretization – convergence theorem

Assumption

Also $F = B$ is linear. (And $A + B$ is maximal dissipative.)

Assumption

- A_h *and* B_h *dissipative*
- $\left\|(A+B)^{-1}v - (A_h + B_h)^{-1}P_h v\right\| \le C(v)h^s$
- $\left\|A_h(A_h + B_h)^{-1}\right\|$ *and* $\cancel{\left\|A_h^2(A_h + B_h)^{-2}\right\|}$ *uniformly bounded*
- ...

Theorem

For $S_h = (I - kB_h)^{-1}(I - kA_h)^{-1}(I + k^2 A_h B_h)$

$$\|u(nk) - S_h^n \eta_h\| \le C(u)(h^s + k)$$

Thank you

E.Hansen, E.Henningsson,
A convergence analysis of the Peaceman–Rachford scheme for semilinear evolution equations,
SIAM J. Numer. Anal., 51 (2013) pp. 1900–1910

E.Hansen, E.Henningsson,
A full space-time convergence order analysis of operator splittings for linear dissipative evolution equations,
to appear in Commun. Comput. Phys. (SCPDE14 special issue)

Extra slides

Full discretization application: dimension splitting

2D diffusion equation

$$\dot{u} = \left(\frac{\partial}{\partial x}\lambda(x)\mu(y)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\lambda(x)\mu(y)\frac{\partial}{\partial y}\right)u, \quad u(0) = \eta$$

with homogenous Dirichlet conditions

Space discretization: quadrature finite element method
Time discretization: Douglas–Rachford

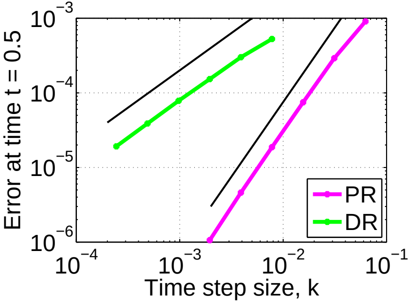
Corollary

$$\|u(nk) - S_h^n\| \leq C(u)(h^2 + k)$$

Full discretization application: dimension splitting

$$\dot{u} = \left(\frac{\partial}{\partial x}\lambda(x)\mu(y)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\lambda(x)\mu(y)\frac{\partial}{\partial y}\right)u$$

$$\lambda(x) = x\sin(\pi x) + 0.1 \qquad \mu(y) = \cos(2\pi y) + 1.1$$



Peaceman–Rachford: $h^2 \sim k^2$, Douglas–Rachford: $h^2 \sim k$